Coupled chaotic maps and propagation of instabilities

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This project was supported by the University of Minnesota's Undergraduate Research Opportunities Program

(1) Description

Generally, a coupled map lattice incorporates a system of equations, a finite number of variables, and iteration of coupling scheme. For the research, it explores the demonstration of the chaotic behavior of the system. First, the recurrence equation is supposed to be set up as $x_j = \mu x_j + d\left(x_{j+1} - 2 * x_j + x_{j-1}\right)$ for each iterator j in the valid domain. For each trail of iteration, the generating function is defined as y=(μ*y+γ*y^3)/(1+β*y^4)+D*y where μ, γ, β, d are real coefficients and D is the system of recurrent equations. Initial condition is default as the generating function starts at 0.01. Meanwhile, number of iteration (N) and dimension of each component of x (T) are fitted as well.

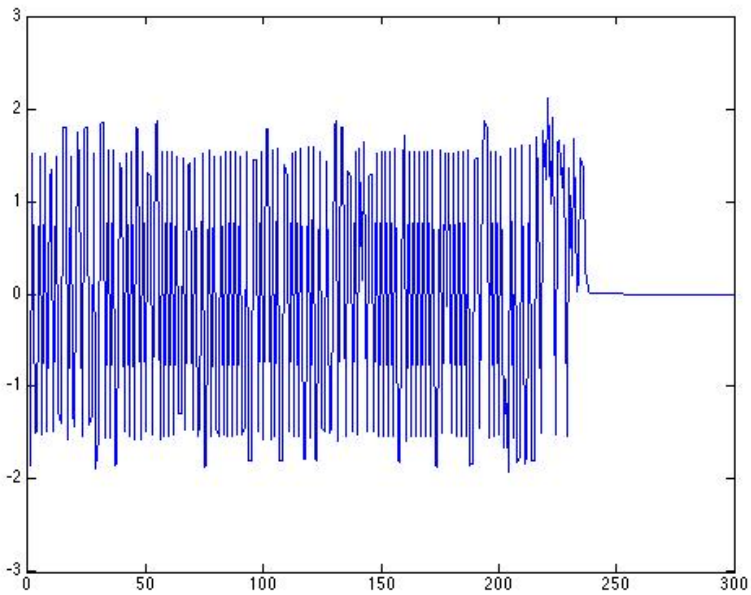


Figure: plot of the chaotic system where N = 800, T = 600, mu=1.7, gamma=0.11, beta=-0.1 and d=0.1

The example shows that zero is unstable. However, to explore how "chaotic" the individual maps are, we sum up heaviside(abs(y)-0.00001) for each y to keep track of the positions.

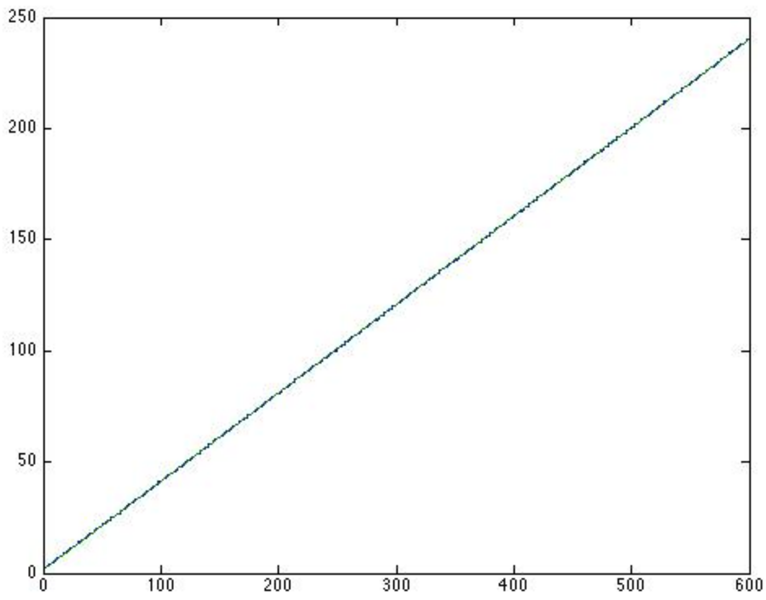


Figure: plot of positions vs linear fit of the positions
The line does fitted the each step of positions

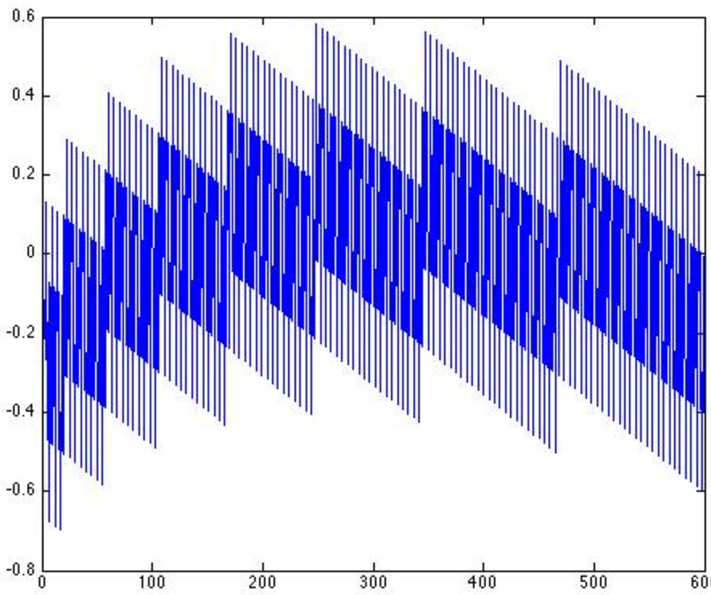


Figure: plot of the difference between positions and the linear fit
The differences are ranged in a certain regime with patterns.

(2) Linear speed

Recalling the setup developed before, for each step j from 1 to N, $x_j = \mu x_j + d\,(x_{j+1} - 2 * x_j + x_{j-1})$ when d = 0.1 and μ = 1.8 at this trail. Referred to characterization of exponential solutions in the linear dynamic by Wim Van Saarloos, currently professor of Theoretical Physics at Leiden University and Director of the Lorentz Center, x can be expressed as $x = e^{\lambda n + \nu j}$ where n represents the time of iteration operating and j is responsible for the each dimension of the space. Substituting the expression into the model for each j of x, we get $e^{\lambda(n+1)}e^{\nu j} = \mu e^{\lambda n+\nu j} + d(e^{\lambda n+\nu(j+1)} - 2e^{\lambda n+\nu j} + e^{\lambda n+\nu(j-1)})$. Dividing by the common factor $e^{\lambda n}e^{\nu j}$ in both left and right side, then we get $e^{\lambda} = \mu + d(e^{\nu} - 2 + e^{-\nu})$. Consider the comoving frame λ is approaching to $\lambda - c\nu$ and then the expression above changes to $e^{\lambda - c\nu} = \mu + d(e^{\nu} - 2 + e^{-\nu})$, which is defined as a system of c with parameters λ and ν. In short, the system can be nominated as $D_c(\lambda, \nu)$. To solve for double root for this system, two criteria should be required: $D_c(0, \nu) = 0$ and $\partial_\nu D_c(0, \nu) = 0$. For real variables c and ν, we solve c in terms of ν: $c = -\frac{d(e^{\nu}-e^{-\nu})}{\mu + d(e^{\nu}-2+e^{-\nu})}$ and also Plugging this backward to solve ν that is defined as the linear spreading speed. Solving in Mathematica, ν = 0.4154821844532788 theoretically.

(3) Pushed versus pulled fronts

For gamma small, precisely for the linear spreading speed, gamma<gamma_*, the speed is independent of gamma. This is the regime of pulled front propagation. The speed of propagation only depends on the linear part of the system and is referred to as the linear speed. For gamma> gamma_*, the speed does depend on gamma. It corresponds to a pushed front.

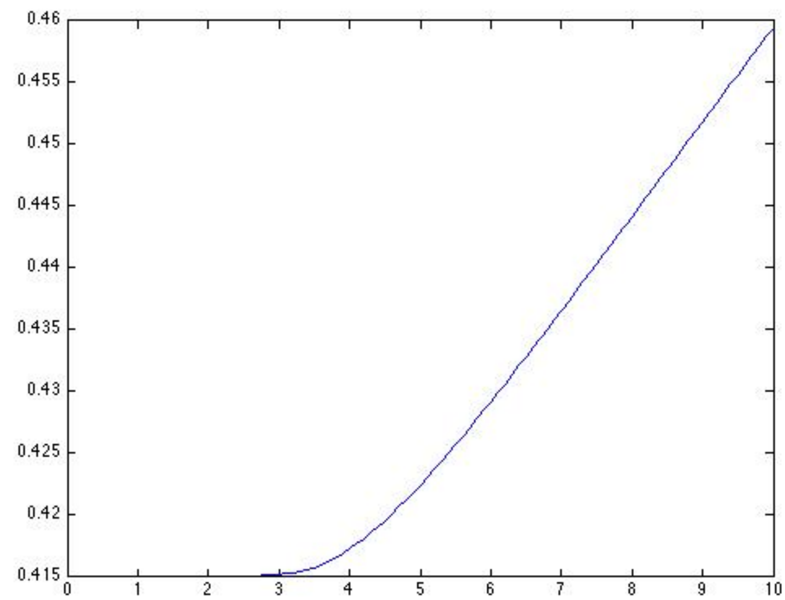


Figure: plot of spreading speed when varying gamma from 0 to 10 (default conditions: d = 0.1 and μ = 1.8).

From the plot, we can roughly estimate that the gamma_* is close to 3. When gamma is smaller than the critical value, the speed keeps 0.415 constantly where as it speeds up when gamma is larger than gamma_*.

(4) Statistics and fluctuations

It is a brief introduction of the observations that the front position is quite deterministic in the pulled regime (small gamma) but behaves roughly like linear + Brownian motion in the pushed regime (large gamma) (see part 8 for further explorations).

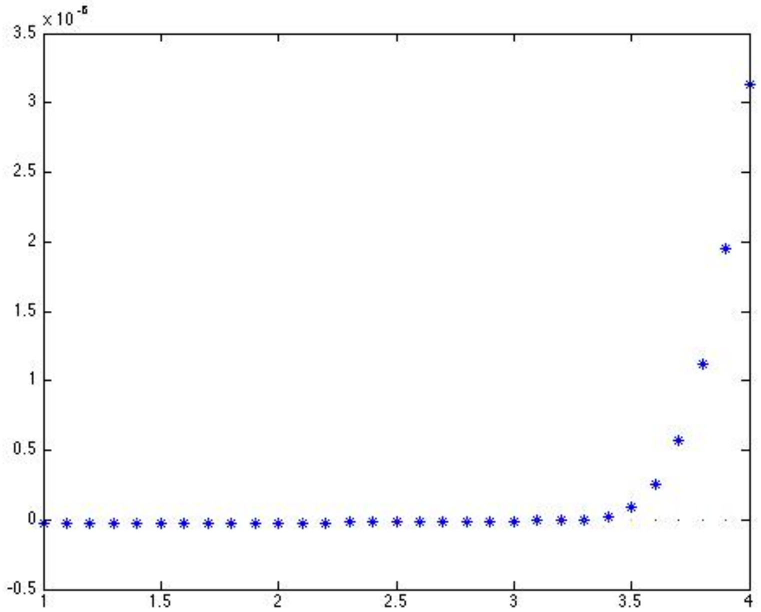


Figure: plot square of deviations from linear speed for pulled front (small gamma)

From the plot, we can see the square of deviations is keeping zero in time.

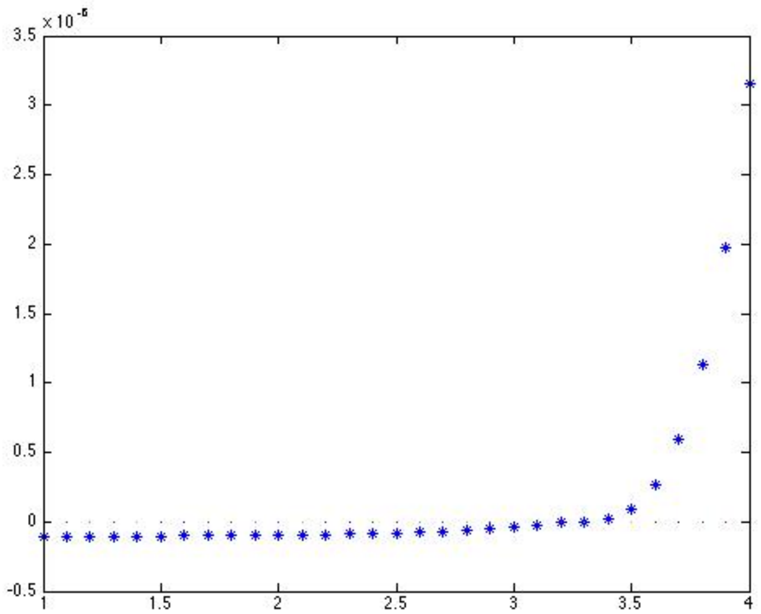
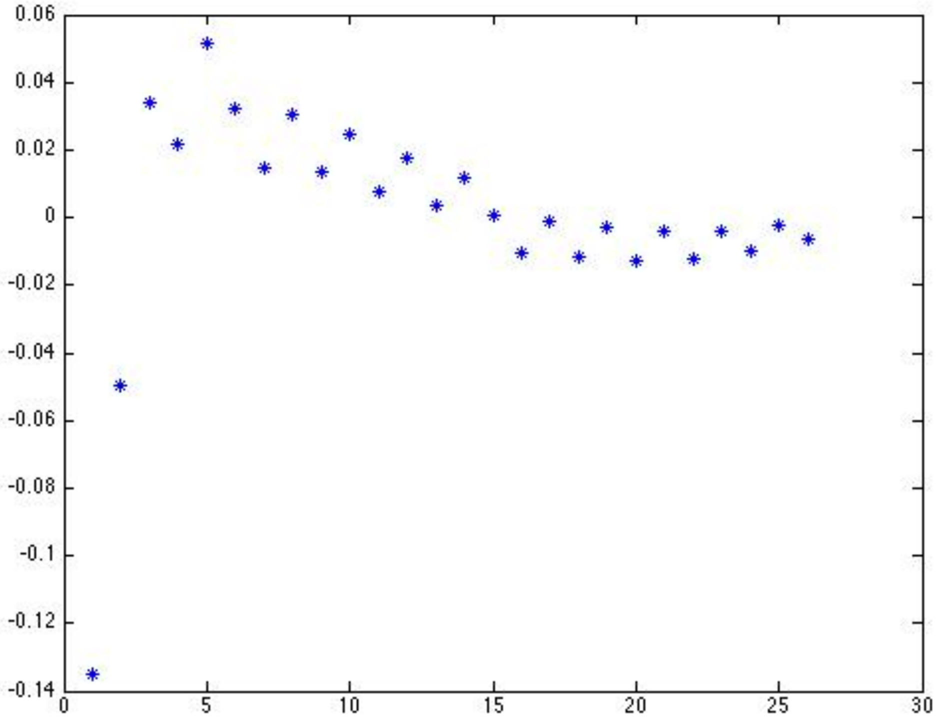
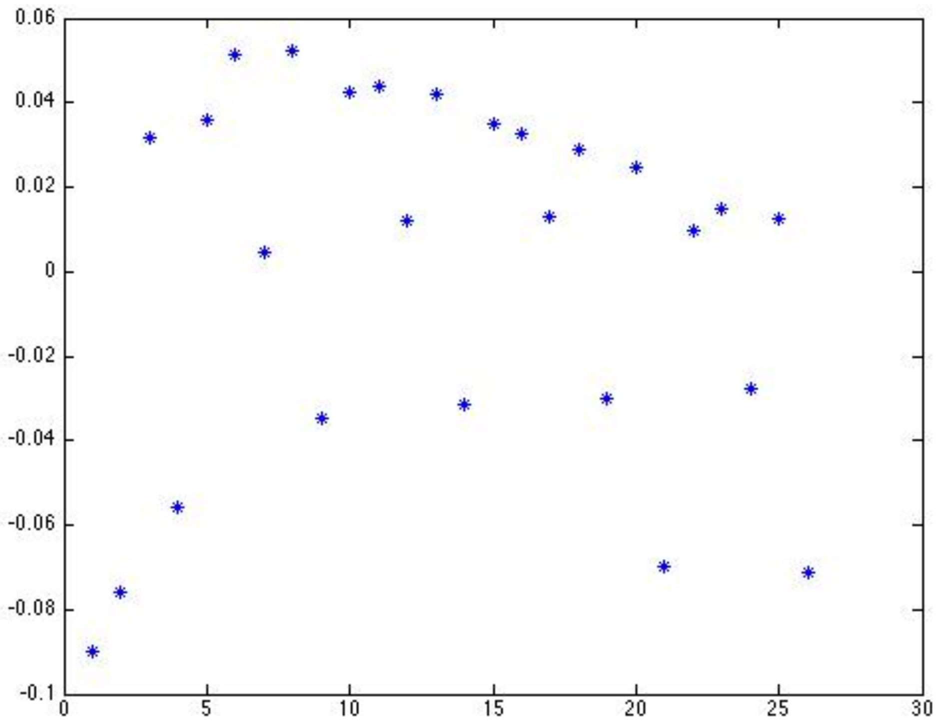


Figure: plot square of deviations from linear speed for pushed front (large gamma)
From the plot, we can see the speed is growing linear roughly in time.

From the other hand, when gamma = 3, we plot the difference between the positions and linear fit



While in gamma = 4.5, the difference is widely distributed, which implies that fluctuations become larger.



According to conjecture that the square of deviations from linear speed is growing linear in time, plots below corroborate the observations that fluctuations are Brownian motion like.

(5) Computing linear speeds numerically

It shows that the speeds converge but quite slowly for small gamma. Therefore, the best strategy to compute the spreading speed is to measure the speed after a long transient, only taking samples from the position at lager time because the speed will be measured more accurately when meeting the convergence. As a result of that, larger time interval (N = 7200) and space interval (T = 5401) should be concerned. For each trail of gamma, we only consider two iterations: when iterator k = T-deltat and k = T where deltat represent the long transient of time (eg. deltat = 1000). Predicted position is defined as sum of Heaviside function from the value of generating function to some designed error (eg. 10^-6). Moreover, we fit an exponent of the leading edge of the position: the we select points from position-5 to position+14 and then do the linear fit of the logarithm of these points. Therefore, the fitted line measures the designed position. In a word, each step of gamma contains two corresponding designed positions. The speed is defined to be the slope of these two positions.

Same algorithm takes actions for all small gammas. For example, when the gamma range varies form 0 to 3.5 and each step = 0.1, the speed roughly keeps constant when gamma is small enough while it increases rapidly after some critical value of gamma. In addition, the trajectory starts to incline after gamma is reaching certain value around 3 (see plot below).

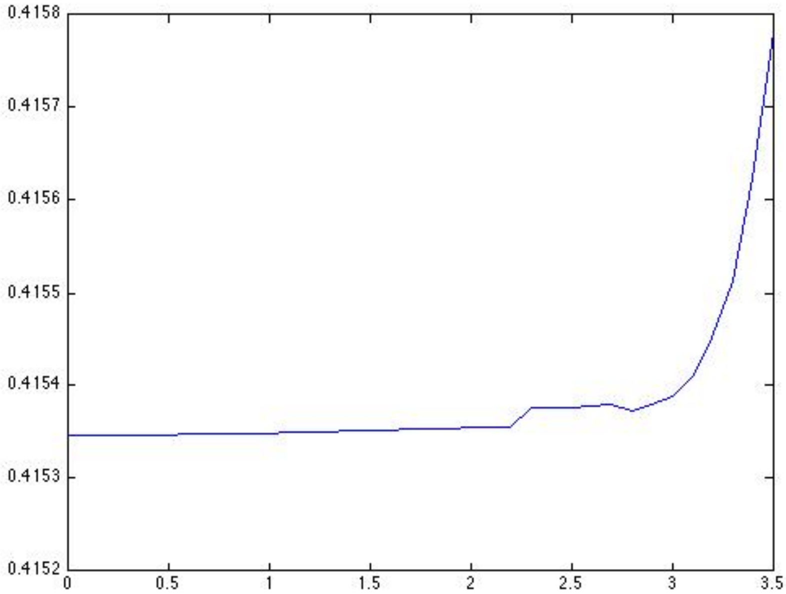


Figure: plot of speed when gamma interval is ranged from 0 to 3.5 (step = 0.1)
Averaging the first 20 speed and we get speed = 0.415349355440927 which is close to the theoretical speed =0.4154821844532788 that recorded in part (2) with error in 10^(-4).

To even longer transient (delta = 2500), the plot is almost reduced extreme fluctuations.

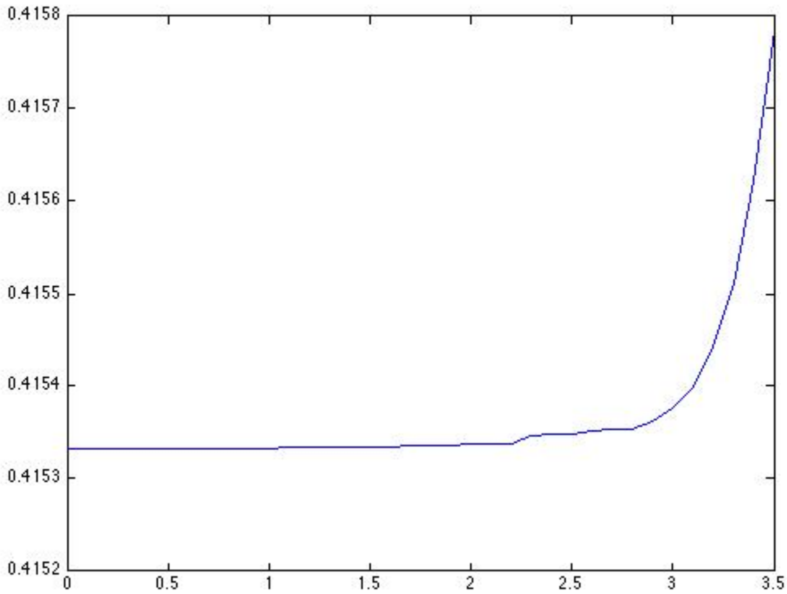


Figure: plot of speed when gamma interval is ranged from 0 to 3.5 (step = 0.05)
Averaging the first 60 speed and we get speed = 0.415338705787356 which is close

to the theoretical speed =0.4154821844532788 that recorded in part (2) with error in 10^(-4). In addition, we figure out that speed = 415472933535917 when gamma = 3.25, which is the closest value to the theoretical speed.

(6) Computing pushed front speeds numerically

Same initial conditions as computing pulled front speeds numerically (N = 7200, T = 5401, deltat = 1000). The gamma range is defined from 3 to 6 and each step is 0.1. Since there are exiting large fluctuations when gamma is larger, our strategy is to use short transient and then average with a linear fit over long time intervals within many points to eliminate fluctuations and estimate the speed as accurate as possible. In each subinterval of gamma, we only guess the front positions when iterator k<T-deltat and mod(k,50) = 0, which indicates that only positions in the valid transient and that are multiples of 50 will be considered. We consider the same following steps as computing pulled fronts: guessing the position, fitting an exponential to the leading edge and then computing the designed positions for each step gamma. Rather than two positions that correspond to each gamma, more positions take account. After that, we do the linear fit of the recorded positions and the slope of the linearity. Therefore, we record each slope corresponds to each step of gamma, which is spreading speed.

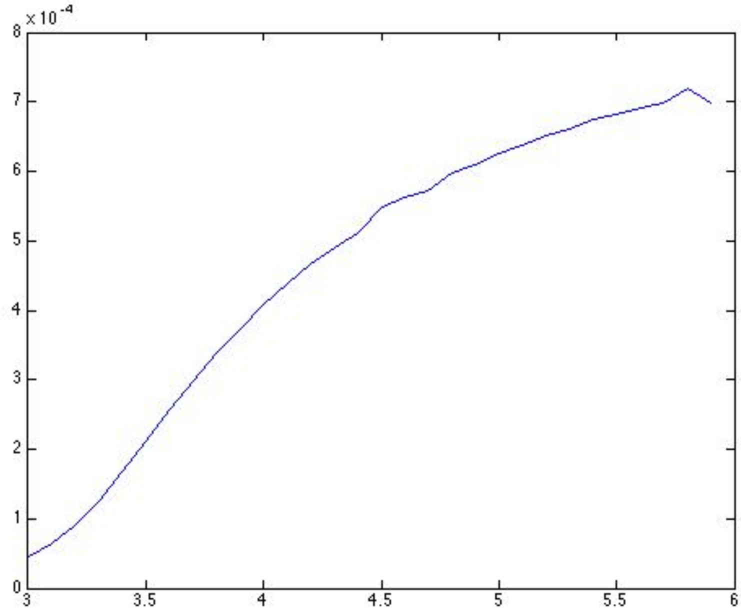


Figure: plot of different from current speed to previous speed due to gamma range [3, 6]

From the plot, speed is increasing with constant rate asymptotically.

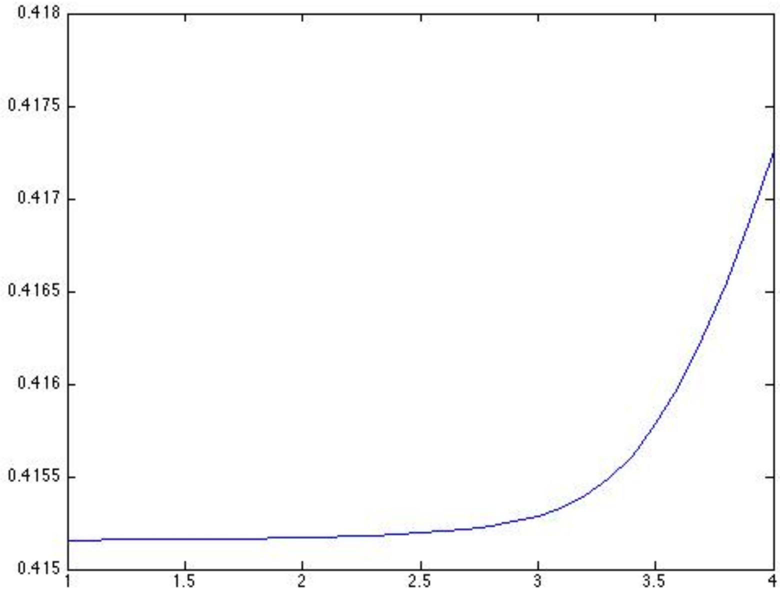


Figure: plot of speed when gamma interval is ranged from 1 to 4 (step = 0.1)

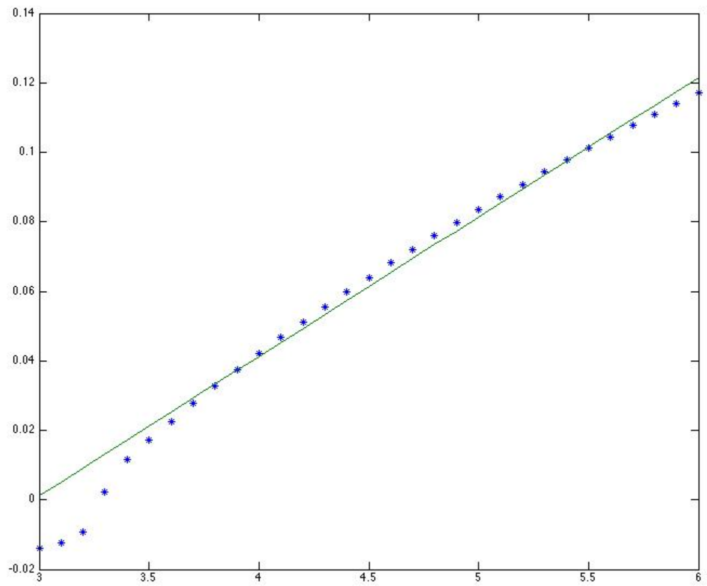


Figure: plot of the squared root of the difference between the computed speed and the theoretical speed (defined in part 2)

If reconsidering the plot above from the fifth speed to the thirty-first one and do the linear fit of them, the slope of the fitted line is 0.040125287172426.

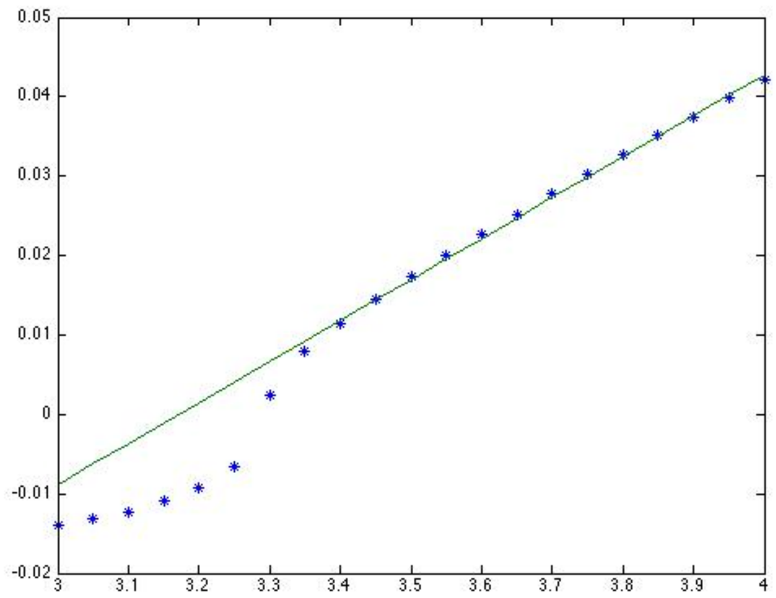


Figure: plot of the squared root of the difference between the computed speed and the theoretical speed (defined in part 2) by varying the gamma range from 3 to 5 (step = 0.05)

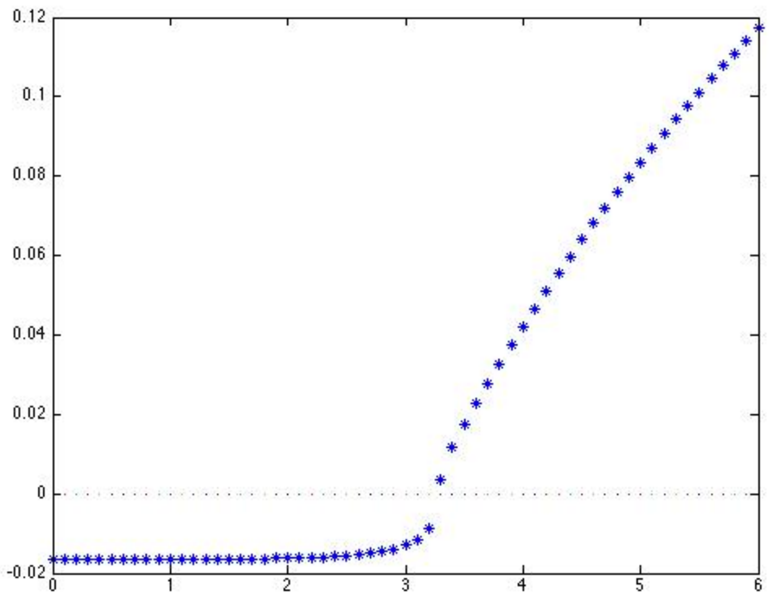
This time, only speeds from the eighth and twenty-first are selected and the slope of the fitted line is 0.051635917759994. Comparing to two consecutive plots, it is obvious that the fitted line overlaps more computed speeds when smaller gamma range is observed.

(7) Estimating the bifurcation value from pulled to pushed

For both pulled front and pushed front, the bifurcation values are analyzed by their best strategies separately (see part (5) and part (6)). To summarize the observations, the squared root of speed – theoretical linear speed should keep constant for small gamma and roughly grow linear for large gamma regime.

If getting speed as a function of time and we only measure every iterator k that is multiple of 50.

It is roughly linear in the pushed regime when gamma is approximately larger than 3.



To fit a line from the thirty-five to sixty-first speed that are all in the pushed regime, we get the slope is 0.040126893771729. From the plot below,

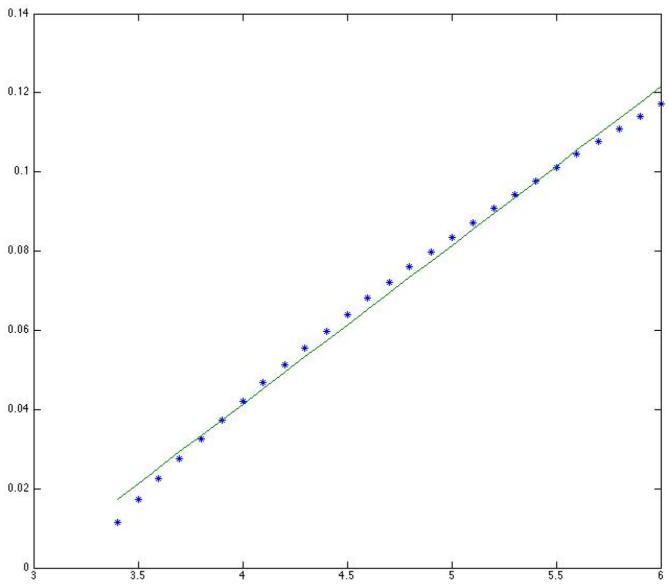


Figure: plot of squared root of the difference between speed and theoretical linear speed and its linear fit.

(8) Fluctuations

According to conjecture that the square of deviations from linear speed is growing linear in time, plots below corroborate the observations that fluctuations are Brownian motion like.

Right after measuring the positions, we do the linear fit of those positions. Moreover, we define the fluctuations are mean squared from positions and the corresponding linear fit. Ranging gamma from 1 to 7 and step = 0.5, we could be able to plot the fluctuations with respect to gamma. When gamma is small (less than 5), the fluctuations are roughly chaotic.

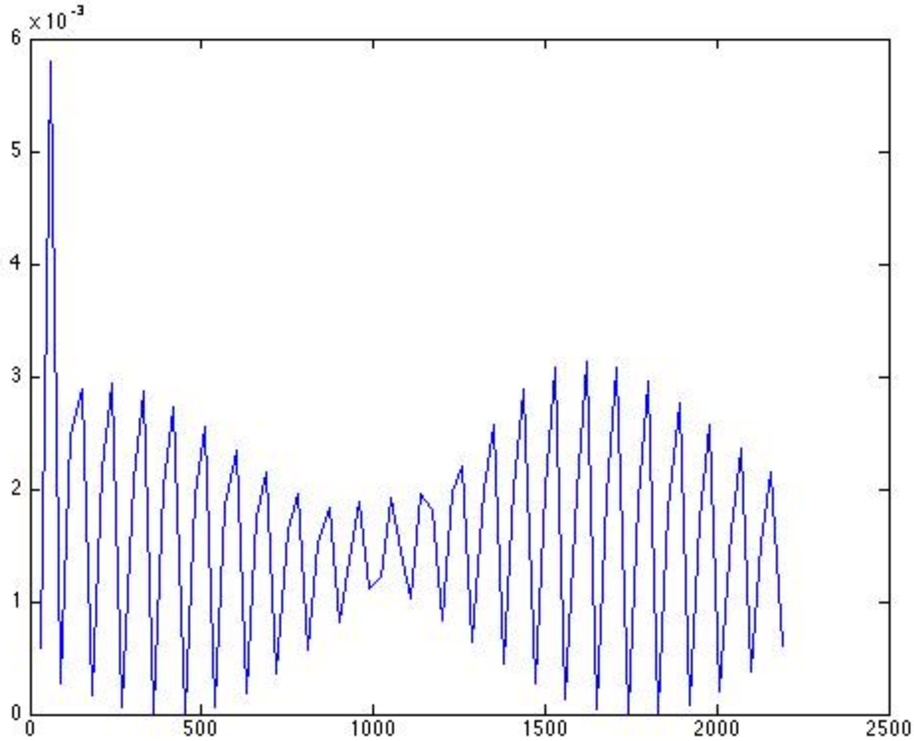


Figure: plot of fluctuations when gamma is reaching at 5.

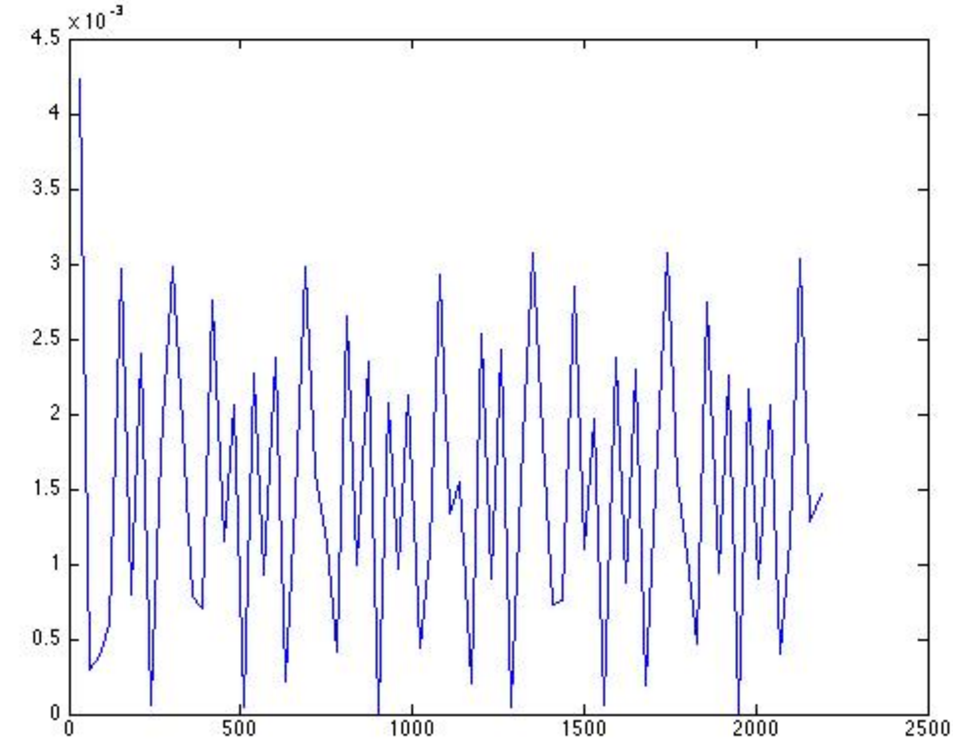


Figure: plot of fluctuations when gamma is reaching at 5.5.

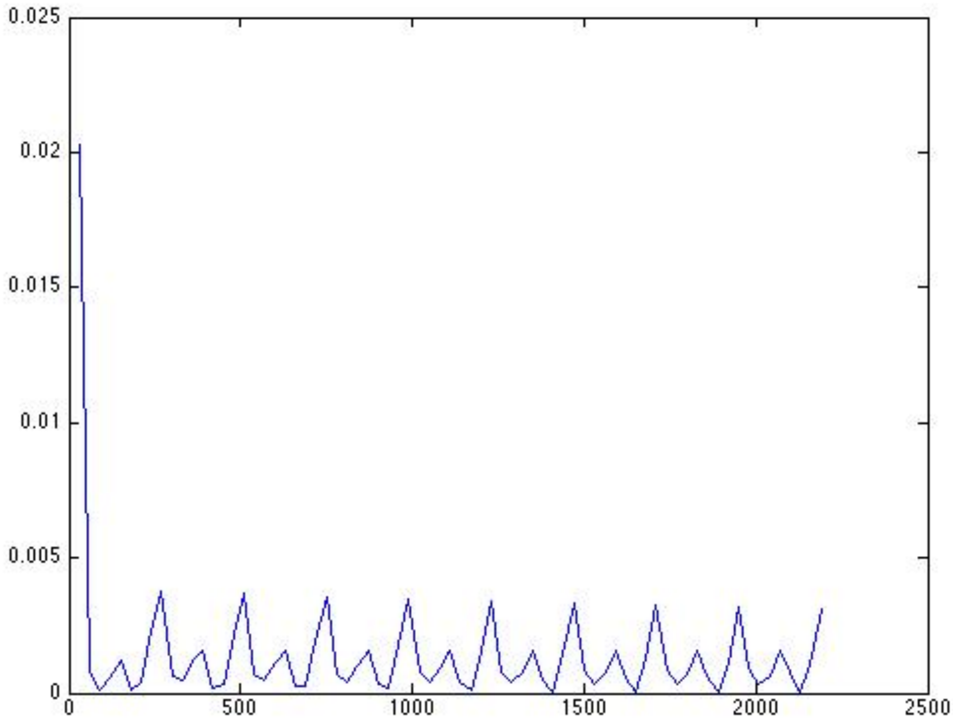
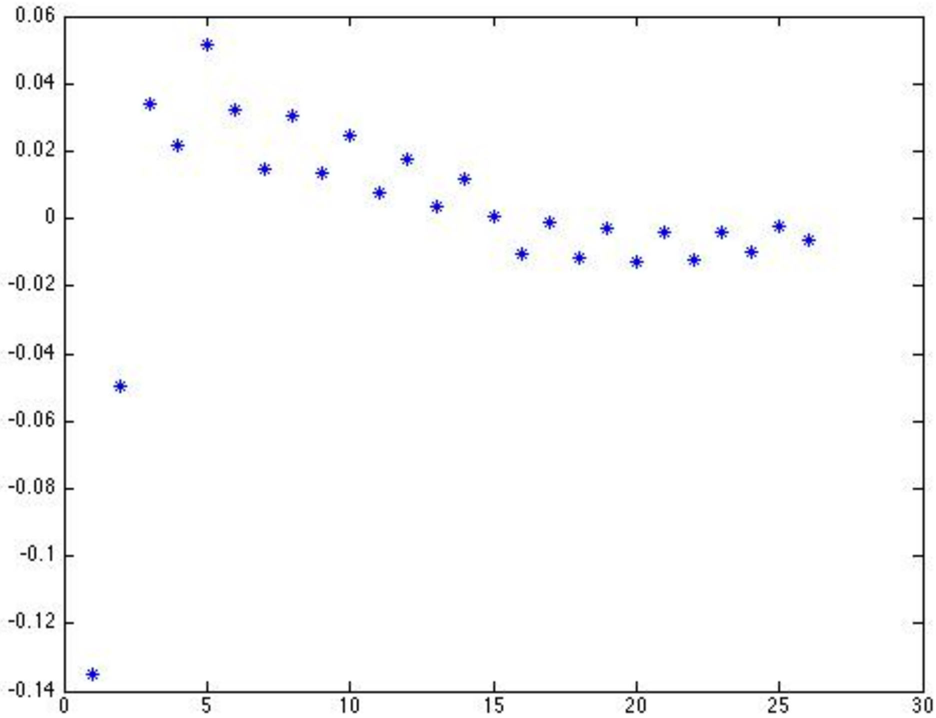
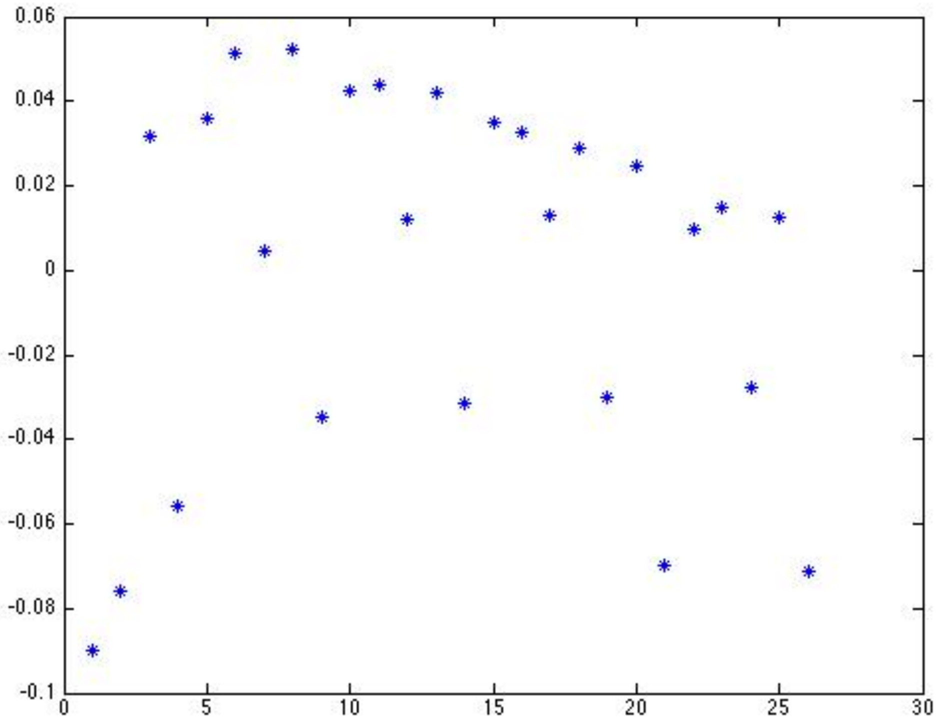


Figure: plot of fluctuations when gamma is reaching at 6.

When gamma = 3, we plot the difference between the positions and linear fit



While in gamma = 4.5, the difference is widely distributed, which implies that fluctuations become larger.



However, when gamma = 6.5, there is a linear trend of the deviations. The fluctuations are grown linear in time according to conjecture.